Problem: 1

Show that C, the set of all non-zero complex numbers is a multiplicative group.

Solution: Let C={z:z=x+iy, x,y ∈ R}. Here R is the set of all real numbers and i= $\sqrt{-1}$.

(1) Closure Axiom: If a+ib ∈ C and c+id ∈ C, then by the definition of multiplication ofcomplex numbers

(a+ib)(c+id)=(ac–bd)+i(ad+bc) ∈ C

Since ac–bd, ad+bc ∈ R for a,b,c,d ∈ R. Therefore,C is closed under multiplication.

(2) Associative Axiom:

(a+ib){(c+id)(e+if)}=(ace–adf–bcf–bde)+i(acf+ade+bce–bdf)

={(a+ib)(c+id)}(e+if) for a,b,c,d ∈ R.

(3) Identity Axiom: e=1(=1+i0) is the identity in C.

(4) Inverse Axiom: Let (a+ib)(≠0)∈C, then

$(a+ib)^{-1}$=1/(a+ib)

=a-ib/$(a^2+b^2)(a+ib)^{-1}$

=a/(a^2+b^2)+i(b/(a^2+b^2)

=m+in ∈ C

where m= a/(a^2+b^2) and n= b/(a^2+b^2). Hence C is a multiplicative group.

Problem: 2

Prove that the set all nth roots of unity with usual multiplication is a group.

Proof:

Let $\boldsymbol{\omega = cos\left(\frac{2\pi}{n}\right) + isin(\frac{2\pi}{n})}$

Then the nth roots of unity are given by $1,\omega,\omega^2,\ldots,\omega^{n-1}$

Let G= $\{1, \omega,\omega^2,\ldots\omega^{n-1}\}$ be the group with respect to multiplication.

We know that $\omega^n = 1, \omega^{n+1} = \omega$ etc.

Let $\omega^r, \omega^s \in G$. Let r+s=qn+t where $0 \le t < n.$ $\omega^r, \omega^s = \omega^{r+s} = \omega^{qn+t} = (\omega^n)^q\omega^t = \omega^t \in G$

We know that usual multiplication of complex number is associative.

$1 \in G$ is the identity element.

Inverse of $\omega^r is\ \omega^{n-r}$.

Hence G is agroup.

Hence the set all nth roots of unity with usual multiplication is a group.

Problem: 3

Let G denote the set of all matrices of the form $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ where $x \in R^*$. Then prove that G is a group under multiplication.

Proof:

Let $A, B \in G$. Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ $and\ B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$.

Closure Axiom:

$$AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in G.$$

Associative Axiom:

We know that matrix multiplication is associative.

Identity Axiom:

Let $E = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$be such that AE=A.

$$\therefore \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\therefore 2xe = x. Hence\ e = 1/2.$$

Hence $E = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is the identity element of G.

Inverse Axiom:

Let $\begin{pmatrix} y & y \\ y & y \end{pmatrix}$ be the inverse of $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$.

Then $\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$

$$\therefore \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\therefore 2xy = \frac{1}{2}$$

$$Hence\ y = \frac{x}{4}$$

$$\therefore Inverse\ of\ \begin{pmatrix} x & x \\ x & x \end{pmatrix}\ is\ \begin{pmatrix} x/4 & x/4 \\ x/4 & x/4 \end{pmatrix}$$

$$Hence\ G\ is\ a\ group.$$

Problem 6: Let G={0,1,2,3,4,5) be a set. Show that under addition modulo 6 G form a group.

Proof:

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

1. From table it is clear that G is closed under closure property as resulting element again element of set G.

2. Clearly associative law hold in G

3. From second row and second column it is clear that 0 is the identity element of the group.

4. From table it is clear that inverse of every element of G exist in G.

    That is $1^{-1}=5$, $2^{-1}=4$, i $3^{-1}=3$,$4^{-1}=2$, $5^{-1}=1$.

5. Since all the elements are symmetrical about principle diagonal, G is abelian.

    Hence G is abelian group.

Problem 5:

Let G={1,2,3,4,5,6} be a set. Show that under multiplication modulo 7 G form a group.

Proof:

$\times_7$	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

1. From table it is clear that G is closed under closure property as resulting element again element of set G.

2. Clearly associative law holds in G

3. From second row and second column it is clear that 1 is the identity element of the group.

4. From table it is clear that inverse of every element of G exist in G. Since all the elements are symmetrical about principle diagonal, G is abelian.

Hence G is abelian group.

Exercises.

1. Prove that C^* is a group under usual multiplication given by $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$.
2. Let$G = \{a + b\sqrt{2} \ : a, b \in \boldsymbol{Z}\}$. Then prove that G is a group under usual addition.
3. Let G={1,I,-1,-i}. Prove that G is a group under usual multiplication.
4. Let G be the set of all real numbers except -1. Define * on G by a*b=a+b+ab. Prove that (G,*) is agroup.
5. In **R**-{1} we define a*b=a+b-ab. Show that(**R**-{1},*) is a group. Is this group abelian?.